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Exact Solutions to the Dirac Equation for Neutrinos Propagating in a Particular Vaidya Background

Víctor M. Villalba[1,2]

In this paper I show that the solutions of the Weyl equation reported by Christodoulakis *et al.* (*Journal of Mathematical Physics* (1994), **35**, 2430) are not single-valued and therefore are unphysical.

After the publication of the pioneering work of Fock and Ivanenko (1929a,b) a large body of papers have been written discussing relativistic electrons in gravitational fields. Among them, it is worth mentioning the paper by Brill and Wheeler (1957) where the problem of neutrinos in gravitational fields is thoroughly analyzed. Here the authors separate variables in Dirac equation in the Schwarzschild background and obtain that the radial dependence of the Dirac spinor satisfies a coupled system of first order ordinary differential equations whose analytic solution in terms of special or algebraic functions is only possible in the trivial case when the mass term in the Schwarzschild line element is zero. Recently, Christodoulakis *et al.* (1994) have reported analytic solutions of the massless Dirac equation in a particular Vaidya background. The authors also claim that after a particular choice of the mass parameter they are able to solve massless Dirac equation in the Schwarzschild gravitational field. In this paper I show that the ansatz (3.1) with (3.3a) and (3.3b) in Christodoulakis *et al.* (1994) lead to solutions of the Dirac equation that are not single-valued or not normalizable, and therefore the exact solutions obtained in Christodoulakis *et al.* are not physical.

The Vaidya line element has the form

$$ds^2 = (1 - 2m(u)/r)du^2 + 2du dr - r^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{1}$$

The covariant generalization of Dirac equation is (Brill and Wheeler, 1957)

$$i\gamma^\mu(\partial_\mu - \Gamma_\mu)\Psi - m\Psi = 0 \tag{2}$$

[1] Institut für Theoretische Physik, Universität Frankfurt, D-60054 Frankfurt am Main, Germany; e-mail: villalba@th.physik.uni-frankfurt.de.
[2] On leave of absence from Centro de Física, IVIC. Apdo 21827, Caracas 1020A, Venezuela.

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where the gamma matrices γ and the spin connections Γ depend on the election of the local tetrad or Vierbein, e_m^μ. Choosing to work (Christodoulakis *et al.*, 1994) in the local (rotating) tetrad gauge

$$e_0^\mu = ((1-2m/r)^{-1/2}, 0, 0, 0), \quad e_1^\mu = (-(1-2m/r)^{-1/2}, \\ (1-2m/r)^{1/2}, 0, 0) \tag{3}$$

$$e_2^\mu = (0, 0, 1/r, 0), \qquad e_3^\mu = \left(0.0, 0, \frac{1}{r\ \sin\theta}\right), \tag{4}$$

we have that if we adopt (4) the spinor solution of Dirac equation (2) is not single valued (Shishkin and Villalba, 1989; Villalba, 1994):

$$\Psi(\varphi + 2\pi) = -\Psi(\varphi) \tag{5}$$

and it is related to Ψ_c, solution of Eq. (2) in the Cartesian (fixed) gauge, as follows

$$\Psi = S\Psi_c, \quad \text{with } S = \frac{1}{r(\sin\theta)^{1/2}} \exp\left(-\frac{\varphi}{2}\gamma^1\gamma^2\right) \exp\left(-\frac{\theta}{2}\gamma^3\gamma^1\right) a \tag{6}$$

where a is a constant nonsingular matrix $a = \frac{1}{2}(\gamma^1\gamma^2 - \gamma^1\gamma^3 + \gamma^2\gamma^3 + 1)$. From (5) and (6) we have that $\Psi_c(\varphi + 2\pi) = \Psi_c(\varphi)$ and $S(\varphi) = -S(\varphi + 2\pi)$.

Using the matrix representation adopted by Christodoulakis *et al.* (1994) the Dirac equation for left handed neutrinos in the Vaidya metric takes the form

$$\left(-\frac{(1+\sigma^1)}{\Delta^{1/2}}\partial_u + \Delta^{1/2}\sigma^1\partial_r + \frac{m\Delta - \dot{m}r}{2r^2\Delta^{3/2}}(1+\sigma^1) - \frac{mI}{2r^2\Delta^{1/2}} \right. \\ \left. + \frac{\Delta^{1/2}}{r}\sigma^1 + \frac{\hat{K}_{\theta,\varphi}}{r}\right)\begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix} = 0 \tag{7}$$

where $\Delta = 1 - 2m(u)/r$, I is the identity matrix, the dot indicates differentiation with respect to u, and the angular operator $\hat{K}_{\theta,\varphi}$ is

$$\hat{K}_{\theta,\varphi} = \sigma^2\partial_\theta + \frac{\cot\theta}{2}\sigma^2 + \frac{\sigma^3}{\sin\theta}\partial_\varphi \tag{8}$$

It is straightforward to verify that the operator $-i\partial_\varphi$ commutes with $\hat{K}_{\theta,\varphi}$ and therefore with Dirac equation (7). Regarding the eigenvalues of $-i\partial_\varphi\Psi = m_\varphi\Psi$ we have that since Ψ_c is single-valued and $S(\varphi + 2\pi) = -S(\varphi)$, Ψ can be written as $\Psi = \Psi(r, \theta)\ \exp(-iEt + im_\varphi\varphi)$, where

$$m_\varphi = N + 1/2, \quad N = 0, \pm 1, \pm 2, \dots \tag{9}$$

Relation (9) tells us that m_φ is never zero and therefore the separation proposed by the ansatz (3.3a,b) is not valid or leads to non–single-valued spinors when we work in the Cartesian tetrad gauge, a result that is inadmissible, and regrettably the solutions derived in Christodoulakis *et al.* (1994) are nonphysical.

The system of Eqs. (3) and (4) in Christodoulakis *et al.* (1994) can also be obtained from (7) and (8) imposing that $\hat{K}_{\theta,\varphi}\Psi = 0$. The angular operator $\hat{K}_{\theta,\varphi}$ does not have zero as eigenvalue. A zero eigenvalue gives as a result a nonnormalizable wave function (Davydov, 1965; Schluter *et al.*, 1983), therefore no physical solutions of the massless Dirac equation can be obtained from Eq. (7).

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